Biking Without Pedaling

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Abstract

It is well known that skilled bicycle riders can balance and propel themselves forward using motions of the handlebar. We present the complete nonlinear dynamics and control of such a pedal-less bicycle with a rider. Propulsion is achieved not by pedaling but by a cyclic motion of the steering axis of the bicycle. It is shown that this kind of actuation results in net forward motion of the bicycle and a building up of momentum. The dynamics of the bicycle and rider in a transverse plane are similar to that of a two link underactuated system where only the second link is actuated. A linear analysis of the bicycle is used to derive a control law for the rider to stabilize the bicycle about its upright position while the periodic motion of the steering axis drives the bicycle forward. After the bicycle attains a higher speed, it is easily stabilizable.

1 Introduction

The dynamics of a bicycle have been widely studied for more than two centuries. The first published study of the dynamics of an uncontrolled upright bicycle was by Whipple [1]. Bicycle dynamics and stability were also studied by, among others, Rankine [2], Sommerfeld and Klein [3], Timoshenko [4], Neĭmark and Fufaev [5], Kane([6], [7]). Weir studied the dynamics of motorcycles and the effect of rider control in his PhD thesis [8]. Motorcycle dynamics was also examined by Sharp [9]. Hand [10] presented a detailed review and comparison of previous work and corrected some of the earlier approaches. More recently, Suryanarayanan et al [11] studied the control of front and rear wheel steered bicycles at very high speeds (80-100 mph). Getz [12] presented a controller to balance and drive a bicycle along a time-parameterized path using a combination of steering and rear-wheel torque. A recent paper by Schwab et al [13] presented benchmark linearized equations for an uncontrolled upright bicycle. Using linear analysis, they showed that uncontrolled bicycles are stable over a certain range of speeds.

While the dynamics and control of bicycles at higher speeds has received a lot of attention, the problem of riding a bicycle at low speeds without pedaling has not yet been adressed. In this paper we will show that a bicycle can be propelled forward from rest without pedaling using only a periodic motion of the steering handle bars. We will also show that the rider can balance the bicycle during the course of this motion. Once the bicycle reaches a higher speed, the self-stabilizing property of bicycles at higher speeds allows it to be easily stabilized. Our motivation for studying this problem comes from the observation that some bicycle riders can often balance almost at rest without pedaling (this can frequently be seen at traffic stops). They achieve this by small adjustments to the steering axis of the bicycle and their body positions during the course of which the bicycles also move by small amounts.

Our interest in this problem was also motivated by our recent work with the Rollerblader([14, 15, 16]).

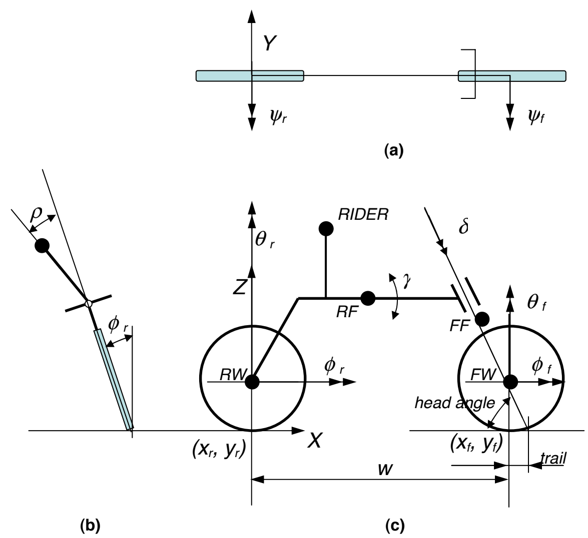


Figure 1: THE BICYCLE MODEL

In recent years there has been considerable interest in similar systems like the Snakeboard [17], the Roller Racer [18] and the TRIKKE [19]. In contrast to more conventional locomotion using legs or powered wheels, these robots rely on relative motion of their joints to generate net motion of the body. The joint variables or *shape* variables, are moved in cyclic patterns giving rise to periodic shape variations called *gaits*. The motion of these systems is the result of a complex interplay between the shape inputs and the nonholonomic constraints acting on the system.

Two systems that are most relevant to this work are the Roller Racer and the TRIKKE. The Roller Racer[1] is a commercial system that can be driven by a single periodic input applied to the steering axis. In [18] Krishnaprasad and Tsakiris analyzed this novel locomotion system and presented analytical and experimental results that showed that the primary source of propulsion for this system was the periodic motion of the steering axis. The TRIKKE[2], is a three-wheeled system produced by **Trikke Tech Inc.**. Analytical and

[1] http://www.ics.forth.gr/~tsakiris/Projects/RR.html

[2] http://www.trikke.com

experimental results for a simplified model of the TRIKKE were presented in [19]. The TRIKKE's method of propulsion is a variant of the Roller Racer's. The main source of propulsion is by periodic motion of the steering handle. In addition, the TRIKKE allows the rider to lean from side to side and use his body weight to speed up the system.

A TRIKKE has three wheels and thus is easy to balance. In contrast, a bicycle, with only two wheels, is an inherently unstable system. Any analysis for the bicycle must take into account the problem of *balancing* the bicycle as well. Rider effects were mostly ignored in the analysis of the Roller Racer but will play a big part in the analysis of the bicycle. A rider is essential to balance the bicycle at lower speeds and we will present a controller that allows the rider to sway from side to side to balance the bicycle.

This paper is organized as follows. In Section 2, we present a detailed and complete nonlinear model for the bicycle. These equations have been derived earlier but mostly with the intention of calculating the resultant linearized equations. Thus, the derivations tend to drop higher order terms from the beginning itself. We present here a complete nonlinear derivation using Lagrange D'Alembert equations. We treat the bicycle as a combination of two rolling disks with the front and rear frames attached to the front and rear disks respectively. Our approach greatly simplifies the derivation of the equations of motion for the system. In Section 3, we examine controllers for maintaining balance for a bicycle with a rider on it. In Section 4, we present the main result for this paper, a method to drive the bicycle forward from rest using only the periodic motion of the steering handle bars.

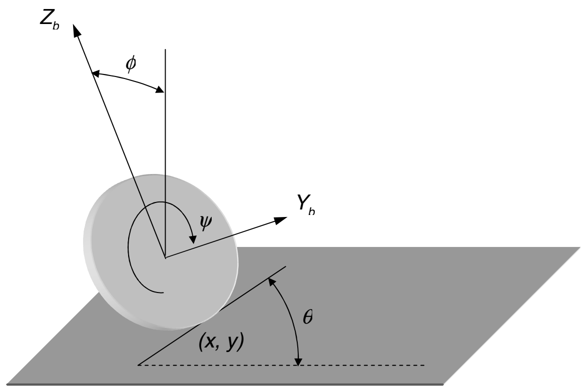


Figure 2: A ROLLING, FALLING DISC

2 Nonlinear Model of the Bicycle

We choose to model the bicycle (Figure 1) as a set of 5 rigid bodies - the front and rear wheels, the front and rear frames and the rider attached by rotary joints. The front and rear wheels of the bicycle are represented as rolling falling disks.

2.1 Rolling, falling disc

We start with a model for a rolling and falling disk as shown in Figure 2. The generalized coordinates required to describe the disk are $(x, y, \theta, \phi, \psi)$. Here, (x, y) represent the coordinates of the contact point of the disk in a global reference frame A. θ represents the angle the disk makes with the positive x axis, ϕ represents the lean or roll angle of the disk, *i.e.* the angle a line joining the contact point on the ground and the highest point on the disk makes with the positive z axis. ψ represents the net rotation of the disk about the body fixed Y_b axis as shown in Figure 2. The generalized coordinates required to describe the two wheels are given by $(x_f, y_f, \theta_f, \phi_f, \psi_f)$ and

$(x_r, y_r, \theta_r, \phi_r, \psi_r)$ where the subscripts f and r represent the front and rear wheels respectively. The rear frame of the bicycle is attached by a rotary joint to the center of the rear wheel. Let γ represent the pitch of the rear frame with respect to the rear wheel. $\gamma = 0$ when the bicycle is in its equilibrium position.

The front frame of the bicycle is attached to the rear frame by a steering axis tilted backwards at an angle α. Let δ represent the angle through which the handle bars are rotated. $\delta = 0$ when the bicycle is in its upright equilibrium position. The rider is modeled as a rotary link attached to the rear frame that can rotate about the body fixed x axis of the rear frame.

Thus, the bicycle (with rider) can be completely described by the set of generalized coordinates $q = (x_r, y_r, \theta_r, \phi_r, \psi_r, \gamma, \delta, \rho, x_f, y_f, \theta_f, \phi_f, \psi_f)$. Let $n = 13$ denote the number of generalized coordinates. The set of parameters needed to describe the bicycle are presented in Table 1. Most of these parameters are based on the data in [13]. All inertia parameters are specified with respect to a body fixed reference frame fixed at the center of mass of the system. All coordinates (for centers of mass, etc.) are specified with the bicycle in its upright equilibrium position and with respect to the Frame XYZ in Figure 1.

2.2 Lagrangian

To derive the complete model for the bicycle we make use of twist vectors and the product of exponentials formulation. (See [20] for an introduction to twists and the product of exponentials formulation). The model for the bicycle can be represented as a series of translations and rotations along the axis given in Table 2.

We can now use the product of exponentials formula to represent the position of the center of mass of the rear wheels and the rear frame in terms of the twist variables (which are also the generalized coordinates representing the rear wheel and rear frame). For example, the position and orientation of the rear wheel is

Variable	Value	Description
w	1.02 m	wheel base
t	0.08 m	trail
α	atan(3)	head angle
R_{rw}	0.3	rear wheel radius
m_{rw}	2 kg	rear wheel mass
x_{rf}	0.3 m	x position of COM of rear frame
y_{rf}	0 m	y position of COM of rear frame
z_{rf}	0.5 m	z position of COM of rear frame
m_{rf}	15 kg	mass of the rear frame
x_{ff}	0.9 m	x position of COM of front frame
y_{ff}	0 m	y position of COM of front frame
z_{ff}	0.7 m	z position of COM of front frame
m_{ff}	4 kg	mass of the front frame
R_{fw}	0.35 m	front wheel radius
m_{fw}	3 kg	front wheel mass
x_{ri}	0.3 m	x position of COM of rider
y_{ri}	0 m	y position of COM of rider
z_{ri}	1.2 m	z position of COM of rider
m_{ri}	40 kg	mass of the rider
x_{le}	0.3 m	x position of rider hinge
y_{le}	0 m	y position of rider hinge
z_{le}	0.7 m	z position of rider hinge

Table 1: GEOMETRIC PARAMETERS FOR THE BICYCLE

represented by

$$g_{rw} = e^{\hat{\xi}_1 x_r} e^{\hat{\xi}_2 y_r} e^{\hat{\xi}_3 \theta_r} e^{\hat{\xi}_4 \phi_r} e^{\hat{\xi}_5 \psi_r} g_{rw}(0) \tag{1}$$

where $g_{rw}(0)$ represents the position and orientation of a reference frame attached to the center of the rear wheel when the bicycle is in its equilibrium position. The front wheel of the bicycle can be represented using a similar procedure and the corresponding generalized coordinates and axes. We can write similar equations for the rear frame(g_{rf}) and the front frame (g_{ff}) of the bicycle.

$$g_{rf} = e^{\hat{\xi}_1 x_r} e^{\hat{\xi}_2 y_r} e^{\hat{\xi}_3 \theta_r} e^{\hat{\xi}_4 \phi_r} e^{\hat{\xi}_6 \gamma} g_{rf}(0),$$

$$g_{ff} = e^{\hat{\xi}_1 x_r} e^{\hat{\xi}_2 y_r} e^{\hat{\xi}_3 \theta_r} e^{\hat{\xi}_4 \phi_r} e^{\hat{\xi}_6 \gamma} e^{\hat{\xi}_7 \delta} g_{ff}(0).$$

ξ	**Twist Vector**	q_i	**Description**
ξ_1	$\begin{bmatrix} 1 & \mathbf{0}_{1\times 5} \end{bmatrix}$	x_r	translation along X.
ξ_2	$[0, 1, \mathbf{0}_{1\times 4}]$	y_r	translation along Y.
ξ_3	$[\mathbf{0}_{1\times 5}, 1]$	θ_r	orientation of rear wheel.
ξ_4	$[\mathbf{0}_{1\times 3}, -1, \mathbf{0}_{1\times 2}]$	ϕ_r	roll angle of rear wheel.
ξ_5	$[R_{rw}, \mathbf{0}_{1\times 3}, -1, 0]$	ψ_r	rotation of rear wheel.
ξ_6	$[R_{rw}, \mathbf{0}_{1\times 3}, -1, 0]$	γ	pitch angle of rear frame.
ξ_7	$[0, (w + t)\sin\alpha, 0, \cos\alpha, 0, -\sin(\alpha)]$	δ	Steering angle.
ξ_8	$[1, \mathbf{0}_{1\times 5}]$	x_f	translation along X.
ξ_9	$[0, 1, 0, \mathbf{0}_{1\times 4}]$	y_f	translation along Y.
ξ_{10}	$[\mathbf{0}_{1\times 5}, 1]$	θ_f	orientation of front wheel.
ξ_{11}	$[\mathbf{0}_{1\times 3}, -1, \mathbf{0}_{1\times 2}]$	ϕ_f	roll angle of front wheel.
ξ_{12}	$[R_{fw}, \mathbf{0}_{1\times 3}, -1, 0]$	ψ_f	rotation of front wheel.
ξ_{13}	$[0, -y_{le}, 0, -1, \mathbf{0}_{1\times 2}]$	ρ	lean of rider.

Table 2: TWIST COORDINATES FOR A BICYCLE. ξ INDICATES THE TWIST, q_i INDICATES THE CORRESPONDING TWIST VARIABLE

We will also derive the position of the center of the front wheel and the orientation of its axle in terms of the generalized coordinates used to describe the rear wheel, the rear frame and the steering column. Thus,

$$g_{fw}^{R} = e^{\hat{\xi}_1 x_r} e^{\hat{\xi}_2 y_r} e^{\hat{\xi}_3 \theta_r} e^{\hat{\xi}_4 \phi_r} e^{\hat{\xi}_6 \gamma} e^{\hat{\xi}_7 \delta} g_{ff}^{R}(0),$$

where we have used the superscript R to denote that this expression is derived using the generalized coordinates for the rear frame and steering column.

Given $g \in \mathrm{SE}(3)$ representing the position and orientation of a rigid body, the body velocity of the body is

$$I_{rw} = \begin{bmatrix} 0.06 & 0 & 0 \\ 0 & 0.12 & 0 \\ 0 & 0 & 0.06 \end{bmatrix} \text{kgm}^2,\ I_{fw} = \begin{bmatrix} 0.14 & 0 & 0 \\ 0 & 0.28 & 0 \\ 0 & 0 & 0.14 \end{bmatrix} \text{kgm}^2,\ I_{rf} = \begin{bmatrix} 9.2 & 0 & -2.4 \\ 0 & 11 & 0 \\ -2.4 & 0 & 2.8 \end{bmatrix} \text{kgm}^2$$

$$I_{rf} = \begin{bmatrix} 0.0546 & 0 & -0.0162 \\ 0 & 0.06 & 0 \\ -0.0162 & 0 & 0.0114 \end{bmatrix} \text{kgm}^2,\ I_{ri} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{kgm}^2$$

Table 3: THE INERTIAL PARAMETERS FOR THE BICYCLE. SUBSCRIPTS *rw, rf, ff, fw, ri* REFER TO THE REAR WHEEL, REAR FRAME, FRONT FRAME, FRONT WHEEL AND RIDER RESPECTIVELY

given by $\xi_b = g^{-1}\dot{g}$. The product of exponentials formulation makes the calculation of this velocity easier. For example, the body velocity of the rear wheel is now given by

$$\hat{\xi}_b = \sum_{i=1}^{5} g_{rw}^{-1} e^{-\hat{\xi}_5 X_5} e^{-\hat{\xi}_4 X_4} \dots e^{-\hat{\xi}_i X_i} \hat{\xi}_i \dot{X}_i e^{\hat{\xi}_i X_i} \dots e^{\hat{\xi}_4 X_4} e^{\hat{\xi}_5 q_5} g_{rw}.$$

where $X = \begin{bmatrix} x_r, y_r, \theta_r, \phi_r, \psi_r \end{bmatrix}$. Let ξ_b represent the body velocity in vector form. The kinetic energy of the rear wheel is now given by $T_{rw} = \xi_b^T \tilde{I}_{rw} \xi_b$. where

$$\tilde{I}_{rw} = \begin{bmatrix} m_{rw}\left[\mathbf{I}_{3\times 3}\right] & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & I_{rw} \end{bmatrix}.$$

while the potential energy is $V_{rw} = m_{rw} g g_{rw}(3,4)$. Here $g_{rw}(3,4)$ is the z coordinate of the center of the rear wheel.

Similar calculations are used to derive the kinetic and potential energy of the rear frame, the rider, the front frame and the front wheel. The total Lagrangian for the system is now given by $L = T_{rw} + T_{rf} + T_{ff} + T_{fw} + T_{rider} - (V_{rw} + V_{rf} + V_{ff} + V_{fw} + V_{rider})$.

2.3 Constraints

The bicycle model derived above uses a total of 13 generalized coordinates. However, the bicycle does not have as many degrees of freedom due to the presence of holonomic and nonholonomic constraints.

While we have used five generalized coordinates to represent the front wheel, the position and orientation of the front wheel $(x_f, y_f, \theta_f, \phi_f)$ is completely determined by the position and orientation of the rear frame and the rear wheel and the steering angle. In addition, the pitch of the rear frame (γ) is similarly constrained. Thus, a set of 5 constraints is needed to represent this dependence. A first set of 3 constraints is obtained by equating the position of the center of the front wheel derived through the rear frame with the position of the same point derived through the front wheel. This leads to the following set of equations:

$$f_1 = g_{fw}^R(1,4) - g_{fw}(1,4) = 0,$$

$$f_2 = g_{fw}^R(2,4) - g_{fw}(2,4) = 0,$$

$$f_3 = g_{fw}^R(3,4) - g_{fw}(3,4) = 0.$$

Two further constraints can be obtained by equating the direction of the body fixed y axis of the front wheel obtained using the two approaches:

$$f_4 = g_{fw}^R(1,2) - g_{fw}(1,4) = 0,$$

$$f_5 = g_{fw}^R(3,2) - g_{fw}(3,4) = 0.$$

A set of four nonholonomic constraints also acts on the bicycle and is given by:

$$\omega_1 = \dot{x}_r + R_{rw}\dot{\psi}_r \cos\theta_r,$$

$$\omega_2 = \dot{y}_r + R_{rw}\dot{\psi}_r \sin\theta_r,$$

$$\omega_3 = \dot{x}_f + R_{fw}\dot{\psi}_f \cos\theta_f,$$

$$\omega_4 = \dot{y}_f + R_{fw}\dot{\psi}_f \sin\theta_f.$$

Thus, our approach models the bicycle using 13 generalized coordinates. There are 5 holonomic and 4 nonholonomic constraints acting on the system.

2.4 Lagrange D'Alembert equations

The first step in deriving the dynamic equations of the bicycle is to differentiate the holonomic constraints. This gives rise to a set of 5 equations linear in the velocities of the system. Combining these equations with the four nonholonomic constraints gives a set of 9 equations linear in the velocities that can be written in the form:

$$A(q)\dot{q} = 0. \tag{2}$$

Since there are 13 generalized coordinates for the system, we conclude that there are 4 independent speeds for the bicycle with a rider. These speeds are any set of 4 that can be chosen from the velocities of the system. However, we choose the following four generalized speeds as a set of independent speeds: $(\dot{\psi}_r, \dot{\phi}_r, \dot{\delta}, \dot{\rho})$. These represent the angular velocity of the rear wheel, the roll rate for the rear frame, the steering angle rate and the lean rate of the rider respectively. Let the corresponding set of generalized coordinates be denoted by $q_d = (\phi_r, \psi_r, \delta, \rho)$. Further, let the remaining coordinates be denoted by $q_k = (x_r, y_r, \theta_r, \gamma, x_f, y_f, \theta_f, \phi_f, \psi_f)$.

Then, we can separate Equation 2 into two parts:

$$A_k(q)\dot{q}_k + A_d(q)\dot{q}_d = 0. \tag{3}$$

Thus,

$$\dot{q}_k = -A_k^{-1}(q)A_d(q)\dot{q}_d. \tag{4}$$

This separation is valid as long as A_k is non-singular. We now write Lagrange's equations using a set of Lagrange multipliers, λ_i's as

$$M_{ij}\ddot{q}_j + C^i_{jk}\dot{q}_j\dot{q}_k + N_i(q,\dot{q}) + (A^T(q)\lambda)_i = \tau_i, i = 1,\ldots,n. \tag{5}$$

Here M represents the mass matrix for the system and is given by $M_{ij} = \frac{\partial^2 L}{q_i q_j}$. C^i_{jk}'s represent the Christoffel symbols for the system. $C^i_{jk} = \frac{1}{2}\big(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{kj}}{\partial q_i}\big)$. N represents the remaining terms given by $N_i = \frac{\partial V}{\partial q_i}$. τ is a one-form of actuator forces acting on the system. There are only two actuators on the system,

1. A steering actuator corresponding to the steering angle δ. This corresponds to a rider applying a torque with his hands to turn the handle bars.

2. A rider lean actuator corresponding to the angle ρ. This corresponds to the torque applied at the hip by a human rider on a bicycle to lean his upper body from side to side.

Thus, $\tau = \begin{bmatrix} \mathbf{0}_{1\times 6} & \tau_\delta & \tau_\rho & \mathbf{0}_{1\times 5} \end{bmatrix}$. The set of allowable directions of motion for the system is given by the null space of the matrix A. We can use Eq. 4 to easily derive an expression for the null space of A. Let Γ denote this null space. Thus, the allowable velocities for the system must lie in the space spanned by the column vectors of Γ, *i.e.*, $\dot{q} = \Gamma\dot{q}_d$. Note that Γ will be a 13×4 matrix. We can now write the reduced set of equations for the constrained system by pre-multiplying Eq. 5 by Γ^T. Since the columns of Γ lie in the null space of A, this operation will eliminate the lagrange multipliers from the resultant set of equations.

It is worthwhile examining this process in more detail since it can lead to significant simplification of the resultant equations, especially when the set of equations is computed symbolically. Eq. 5 can be split, after writing it in matrix form, as

$$M_k\ddot{q}_k + M_d\ddot{q}_d + C(q)[\dot{q},\dot{q}] + N(q,\dot{q}) + A^T(q)\lambda = \tau. \tag{6}$$

Here, the notation $C(q)[\dot{q},\dot{q}]$ for the centrifugal and coriolis terms indicates their bilinear dependence on the velocities. Differentiating Eq. 4, we can write

$$\ddot{q}_k = -\frac{\partial\big[A_k^{-1}(q)A_d(q)\big]}{\partial q_k}\dot{q}_k\dot{q}_d - \frac{\partial\big[A_k^{-1}(q)A_d(q)\big]}{\partial q_d}\dot{q}_d\dot{q}_d - A_k^{-1}(q)A_d(q)\ddot{q}_d. \tag{7}$$

Using Eq. 4 to substitute for $\dot{q}_k$, we write

$$\ddot{q}_k = -\frac{\partial\big[A_k^{-1}(q)A_d(q)\big]}{\partial q_k}\big(-A_k^{-1}(q)A_d(q)\big)\dot{q}_d - \frac{\partial\big[A_k^{-1}(q)A_d(q)\big]}{\partial q_d}\dot{q}_d\dot{q}_d - A_k^{-1}(q)A_d(q)\ddot{q}_d. \tag{8}$$

Substituting for $\ddot{q}_k$ in Eq. 6 and then pre-multiplying by Γ^T to eliminate the Lagrange multipliers, we find

$$\tilde{M}\ddot{q}_d + \tilde{C}(q)[\dot{q}_d,\dot{q}_d] + \tilde{N}(q,\dot{q}_d) = \tilde{\tau}. \tag{9}$$

Here,

$$\begin{aligned}
\tilde{M} &= \Gamma^T M_d + \Gamma^T M_k(-A_k^{-1}(q)A_d(q)),\\
\tilde{C} &= \Gamma^T M_k\Big(-\frac{\partial\big[A_k^{-1}(q)A_d(q)\big]}{\partial q_k}\big(-A_k^{-1}(q)A_d(q)\dot{q}_d\big)\dot{q}_d \\
&\quad - \frac{\partial\big[A_k^{-1}(q)A_d(q)\big]}{\partial q_d}\dot{q}_d\dot{q}_d\Big) + \Gamma^T C(q)[\dot{q},\dot{q}],\\
\tilde{N} &= \Gamma^T\frac{\partial V}{\partial q},\\
\tilde{\tau} &= \Gamma^T\tau.
\end{aligned}$$

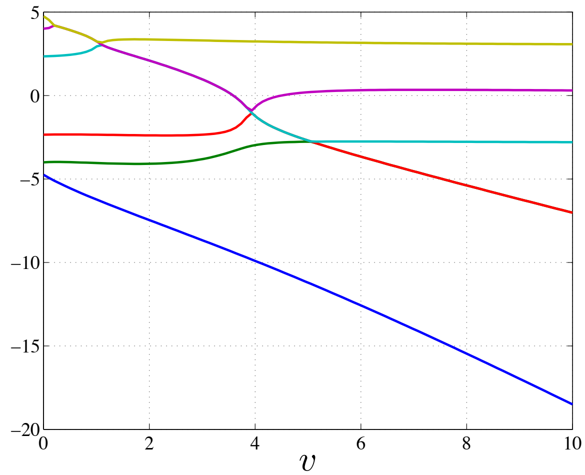


Figure 3: EIGENVALUES FOR LINEARIZED BICYCLE WITH RIDER

$\frac{\partial A_k^{-1}}{\partial q}$ is often difficult to compute symbolically while $\frac{\partial A_k}{\partial q}$ is easier to compute. $\frac{\partial A_k^{-1}}{\partial q}$ may be computed numerically using the following equation:

$$\frac{\partial A_k^{-1}}{\partial q} = -A_k^{-1}\frac{\partial A_k}{\partial q}A_k^{-1}.$$

Equations 4 and 9 together represent the complete reduced dynamics of the system using a reduced set of variables $(q_k, q_d, \dot{q}_d)$.

2.5 Linear Analysis

A linear analysis similar to [13] can be carried out for the Bicycle with a rider. The analysis is carried out about a straight ahead, vertical position of the bicycle with a forward velocity v. It results in a set of equations of the form

$$M_L\ddot{q}_d + \left[C1.v\right]\dot{q}_d + \left[K0 + K2.v^2\right]q_d = B_L\tau_L. \tag{10}$$

where $q_d = (\phi_r, \psi_r, \delta, \rho)$, $\tau_L = (\tau_\delta, \tau_\rho)$ and

$$M_L = \begin{bmatrix} 93.3121 & 0 & -1.18547 & 44 \\ 0 & 6.0857 & 0 & 0 \\ -1.18547 & 0 & 0.2863 & -0.4464 \\ 44.000 & 0 & -0.4464 & 30.0000 \end{bmatrix},$$

$$C1 = \begin{bmatrix} 0 & 0 & -26.2328 & 0 \\ 0 & 0 & 0 & 0 \\ 0.7292 & 0 & 1.4703 & 0 \\ 0 & 0 & -7.0686 & 0 \end{bmatrix},$$

$$K0 = \begin{bmatrix} -588.1095 & 0 & 19.1697 & -196.2 \\ 0 & 0 & 0 & 0 \\ 19.1697 & 0 & -6.0620 & 0 \\ -196.2 & 0 & 0 & -196.2 \end{bmatrix}$$

$$K2 = \begin{bmatrix} 0 & 0 & -55.3864 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2.0528 & 0 \\ 0 & 0 & -18.6016 & 0 \end{bmatrix}$$

Figure 3 shows a plot of the eigenvalues for the open loop response of the system vs. the forward speed v. It can be seen that one of the eigenvalues (corresponding to the uncontrolled rider) is always in the right half plane. Further, two other eigenvalues have positive real parts for all v except for a certain range of values. This is well known for an uncontrolled bicycle without a rider [13]. It is also clear that except for the one eigenvalue corresponding to the uncontrolled rider, the bicycle would be easily controllable at higher speeds since all the other eigenvalues either have negative real parts or (for one eigenvalue) a very small positive real part that can be easily controlled.

Our final goal in this paper is to be able to propel the bicycle without pedaling and simultaneously maintain balance. We will first examine in the next section the latter aim, *i.e.* maintaining balance for the bicycle. We are particularly interested in controllers that can achieve this at lower speeds. We will then examine (in Section 4) the method for driving the bicycle without pedaling.

3 Control of a Bicycle with a Rider

In this section, we will examine controllers for maintaining balance for a bicycle with a rider on it. We will first examine input-output linearizing controllers and show that these controllers are unsuitable for bicycles at rest. We will then look at two controllers derived using linearized models for the bicycle.

3.1 Input Output Linearization

The Bicycle derived above has two inputs - the steering input and the rider lean input. It seems conceivable that we could carry out an input-output linearization of the bicycle to linearize the response of two outputs using the two inputs.

3.1.1 Input-output linearization with output (ϕ_r, ρ)

Choose the output for the bicycle as $z = (\phi_r, \rho)$. Using Eq. 9 and extracting the relevant terms, we can express the evolution of the output in the following form,

$$\ddot{z} = \begin{pmatrix} \ddot{\phi}_r \\ \ddot{\rho} \end{pmatrix} = \begin{pmatrix} E_{\phi_r} \\ E_\rho \end{pmatrix} + F_{(\phi_r,\rho)} \begin{pmatrix} \tau_\delta \\ \tau_\rho \end{pmatrix}. \tag{11}$$

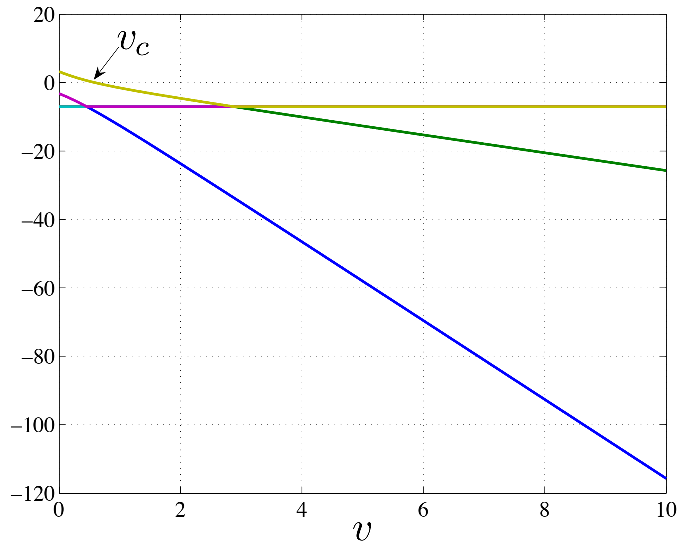


Figure 4: EIGEN VALUES: INPUT-OUTPUT LINEARIZED SYSTEM WITH OUTPUT (ϕ_r, ρ)

Now, a control law to linearize the response of the output $z = (\phi_r, \rho)$ is given by,

$$\begin{pmatrix} \tau_\delta \\ \tau_\rho \end{pmatrix} = -F^{-1}_{(\phi_r,\rho)} \Big[\begin{pmatrix} E_{\phi_r} \\ E_\rho \end{pmatrix} + \begin{pmatrix} v_{\phi_r} \\ v_\rho \end{pmatrix} \Big].$$

Substituting this controller into Eq. 11, we find $\ddot{z} = \begin{pmatrix} v_{\phi_r} & v_\rho \end{pmatrix}^T$. We can now choose v_{ϕ_r} and v_ρ to regulate (ϕ_r, ρ) to the equilibrium position $(0, 0)$.

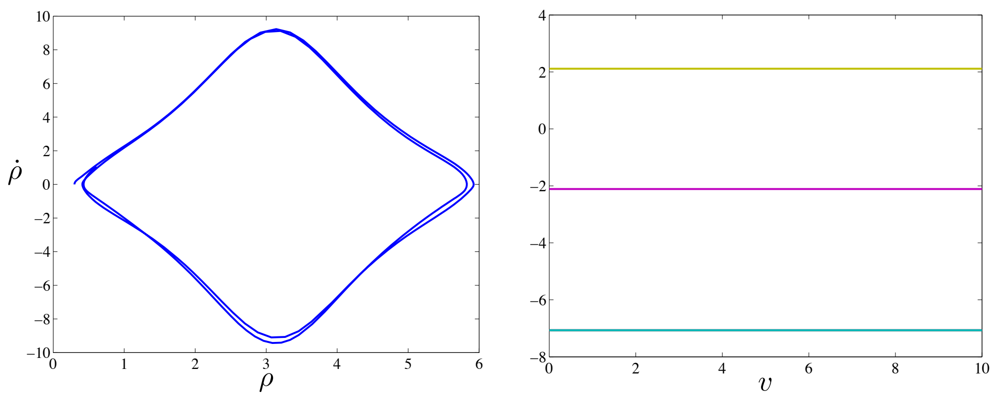
$$\begin{pmatrix} v_{\phi_r} \\ v_\rho \end{pmatrix} = -K_p \begin{pmatrix} \phi_r \\ \rho \end{pmatrix} - K_d \begin{pmatrix} \dot{\phi}_r \\ \dot{\rho} \end{pmatrix}. \tag{12}$$

Here, K_p and K_d are chosen as positive definite matrices such that the resulting error dynamics for (ϕ_r, ρ) exponentially converge to $(0, 0)$. The linearizing controller is substituted back into the dynamic equations for the system and a linear analysis is carried out by linearizing the system about a straight ahead, vertical position of the system with forward velocity v. The eigen values corresponding to this linearized system are plotted versus the forward speed v in Figure 4. The system starts off with an eigenvalue with positive real part at $v = 0$. The real part of this eigenvalue becomes negative at a critical speed $v = v_c = 0.5906$ m/s after which the controller easily stabilizes the system. The critical speed v_c corresponds to a rear wheel angular velocity of approximately -2 rad/s. It should be noted that the imaginary parts of the eigenvalues are very small (of the order of 10^{-6}).

3.1.2 Input-output linearization with output (ϕ_r, δ)

An alternative set of outputs can be chosen as (ϕ_r, δ). Let the desired behavior of the output be defined as $\phi_d = 0, \dot{\phi}_d = 0, \delta_d = 0, \dot{\delta}_d = 0$. Input-output linearization using the choice of output given above results in zero dynamics for the bicycle system. We are particularly interested in the zero dynamics of the rider lean angle, *i.e.* what is the evolution of ρ corresponding to the case where the output is forced to follow its desired behavior exactly? The answer is found by substituting $\phi_r = \phi_d, \dot{\phi}_r = \dot{\phi}_d$, $\delta = \delta_d, \dot{\delta} = \dot{\delta}_d$ into the bicycle dynamics. Figure 5(a) shows a section of the behavior of ρ corresponding to an initial condition of $(\rho, \dot{\rho}) = (0.3, 0)$. A linearized analysis of the zero dynamics for ρ reveals the existence of a center at $(\rho, \dot{\rho}) = (\pi, 0)$, the vertical down position of the rider. Thus, linearizing the response of ϕ_r, δ results in oscillatory motion of the rider about the vertical down position. This behavior is obviously not desired.

Figure 5(b) shows the eigen values for the controller described here. It is easy to see that this controller is always unstable. This behavior, where linearizing the response of the first link of a two-link underactuated systems leads to undesirable zero-dynamics for the actuated second link, has also been observed for the Acrobot model [21].



(a) ZERO DYNAMICS OF BICYCLE : RESPONSE OF ρ

(b) EIGEN VALUES

Figure 5: INPUT-OUTPUT LINEARIZED SYSTEM WITH OUTPUT (ϕ_r, δ)

3.1.3 Input-Output Linearization: Single Output

Motivated by the actual behavior of a bicycle, a single output can also be chosen for input-output linearization. As was shown in [13], an unactuated bicycle is stable over a small range of forward velocity of the bicycle. The difference between the model used in [13] and our model is the presence of the rider and the two actuated inputs. Thus, it may be possible to stabilize the bicycle at a higher speed by using the lean input τ_ρ to regulate the output ρ to $\rho_d = 0$. The basic idea here is to have the rider and bicycle behave as a single rigid body. The resultant system now has properties similar to the system in [13], *i.e.* a range of speed (different from the range for the system in [13]) over which the system is stable.

Choose the output for the bicycle as ρ. We will also use only one input τ_ρ and set $\tau_\delta = 0$. We can gain more insight into the behavior of the system through a linear analysis. Figure 6 represents the real parts of the eigen values of the resulant linearized system after substituting the input-output linearizing controller for ρ. The system starts off with two eigenvalues with positive real parts at zero speed. As the forward velocity

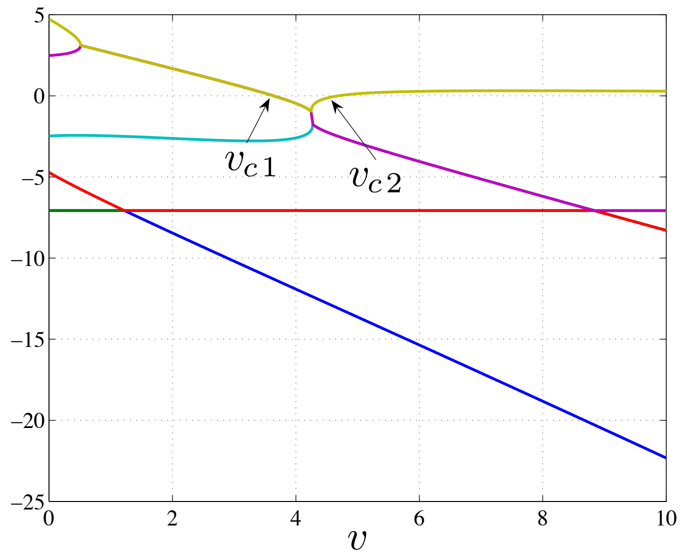


Figure 6: EIGEN VALUES: INPUT-OUTPUT LINEARIZED SYSTEM WITH OUTPUT (ρ)

v of the bicycle increases, these two eigen values coalesce into a pair of conjugate values whose real part becomes negative for $v = v_{c1} = 3.64$ m/s. For $v > v_{c2} = 4.685$ m/s, the real part of one of the eigen values becomes positive again. Thus, for $v_{c1} < v < v_{c2}$ the controller derived with output ρ will be stable.

It is obvious from the results of this section that input-output linearization is a bad choice for control of the bicycle at zero velocity since it results in undesirable zero dynamics. We will now derive a balancing controller for the bicycle at zero speed based on a linear analysis of the system. The controller will be derived using pole-placement for the linear model of the bicycle derived in Section 2.5.

3.2 Balancing Controllers at low speeds

Using the linear model for the bicycle derived in Section 2.5, a linear controller for balancing the bicycle can be easily derived using pole-placement techniques. The poles of the system were placed at

$(-10, -7.5, -6, -5, -3, -2)$. This results in a feedback law of the form $\tau = -KX_r$ where $\tau = (\tau_\delta, \tau_\rho)$ and $X_r = (\dot{\phi}_r, \dot{\delta}, \dot{\rho}, \phi_r, \delta, \rho)$. Also,

$$K = 100 \begin{pmatrix} -2.93 & 0.09 & -1.29 & -7.40 & 0.36 & -2.70 \\ -55.00 & 1.24 & -24.02 & -138.22 & 4.09 & -49.95 \end{pmatrix} \tag{13}$$

4 Forward propulsion of the bicycle without pedaling

We will now present the main result for this paper, *i.e.* the propulsion of the bicycle without pedaling. The propulsion is achieved by commanding the steering axis of the bicycle to follow a desired sinusoidal trajectory.

4.1 Periodic input

We start with the complete nonlinear equations of the bicycle given by Eq. 9. The desired trajectory for the steering angle δ is given by:

$$\delta_d = \delta_a \sin(2\pi \frac{t}{T}).$$

To derive a control law that makes δ follow δ_d, we will linearize the dynamics corresponding to δ. This is achieved using nonlinear feedback involving terms in $(q_k, q_d, \dot{q}_d)$. We assume that these quantities are available to us from different sensors, either encoders or an inertial measurement unit. From Eq. 9, we have

$$\ddot{q}_d + \tilde{M}^{-1}\tilde{C}(q)[\dot{q}, \dot{q}] + \tilde{M}^{-1}\tilde{N}(q, \dot{q}) = \tilde{M}^{-1}\tilde{\tau}. \tag{14}$$

The equation for $\ddot{\delta}$ is given by the 3^{rd} row of Eq. 14:

$$\ddot{\delta} = E_\delta(q_k, q_d, \dot{q}_d) + F_\delta \tau_\delta + F_\rho \tau_\rho. \tag{15}$$

Here E_δ represents all the terms in the equation that are dependent on $(q_k, q_d, \dot{q}_d)$ while F_δ and F_ρ are the coefficients to the steering and rider lean torques respectively.

4.2 Balancing Controllers

We will now illustrate the derivation of the complete control laws using the balancing controller derived in Section 3.2. For the balancing controller derived using pole-placement techniques, the controller torques are:

$$\tau_\rho = -K_{2p}\begin{bmatrix}\dot{\phi}_r & \dot{\delta} & \dot{\rho} & \phi_r & \delta & \rho\end{bmatrix}^T \tag{16}$$

$$\tau_\delta = \frac{1}{F_\delta}\big(K_p(\delta_d - \delta) + K_d(\dot{\delta}_d - \dot{\delta}) - E(q_k, q_d, \dot{q}_d) - F_\rho \tau_\rho\big). \tag{17}$$

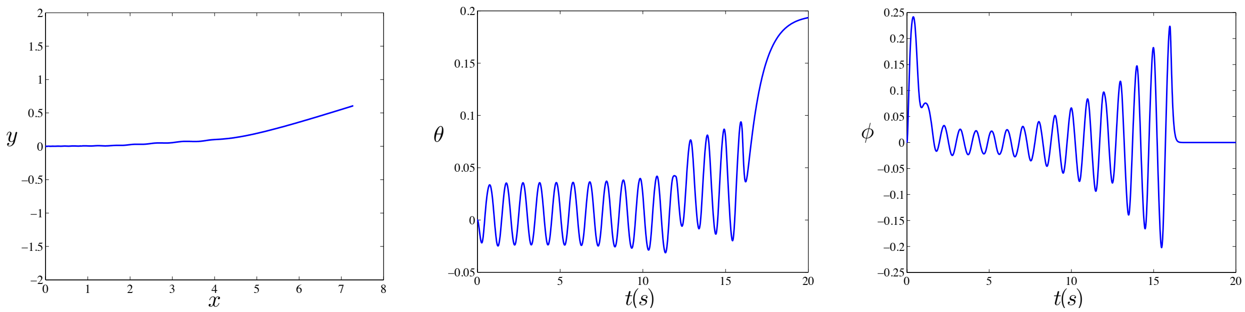
where K_{2p} is the second row of the gain matrix K in Eq. 13.

4.3 Balancing Controller at higher speeds

Once the bicycle has gained speed and is traveling at more than approximately 0.6 m/s, the input-output linearizing controller derived in Section 3.1.1 can be used to stabilize the bicycle.

4.4 Simulation results

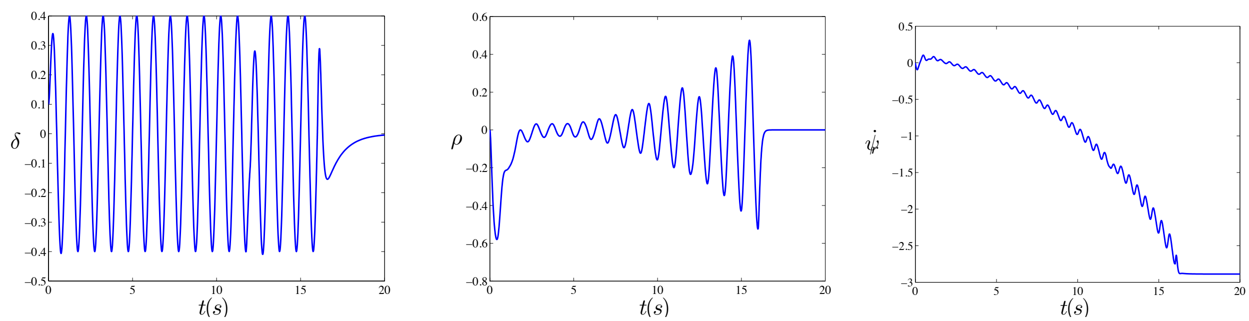
Figure 7 presents results for the case where the pole placement based balancing controller is used during the initial stages. The initial condition for the state of the system is $q = (\mathbf{0}_{4\times 1}, -0.0001, 0, 0.1, 0, w, 0.0075, -0.0949, -0.0316, 0)$ and $\dot{q}_d = (-0.1, 0, 0, 0)$. The system uses the control law described in Section 4.1 for a period of 12 seconds. During this period the steering angle is subjected to a sinusoidal motion with an amplitude of 0.4 and a frequency of 1Hz. The system builds up momentum for 12 seconds. Note that the system also recovers



(a) TRAJECTORY OF CONTACT POINT OF REAR WHEEL

(b) θ VS. t

(c) ROLL ANGLE OF BICYCLE



(d) STEERING ANGLE

(e) LEAN ANGLE

(f) ANGULAR VELOCITY OF REAR WHEEL

Figure 7: SIMULATION RESULTS WITH POLE-PLACEMENT BASED CONTROLLER

from its initial falling roll velocity. Further, although the system deviates considerably from its equilibrium position, the linear controller is still able to balance the bicycle. It should be noted that the bicycle at rest is a very unstable system and may not be able to recover from bigger disturbances. However, as noted earlier a bicycle in motion exhibits better stability.

After 12 seconds, the forward speed of the bicycle is greater than the critical speed v_c required for the input-output controller derived in Section 4.3 to stabilize the system. We now use the input-output linearizing controller derived in Section 4.3 that attempts to regulate the roll of the rear frame (ϕ_r) and the lean of the rider (ρ) to the equilibrium upright position. As can be seen from the system response, the bicycle is easily stabilized at the higher speed. The system finally stabilizes to an upright position with a constant forward velocity.

5 Motion Planning

Once the bicycle has gained some velocity, it can be steered by the rider by leaning from side to side. In this section, we present a rudimentary controller that lets a rider converge to a desired trajectory. The controller functions by exploiting a phenomenon commonly referred to as *counter-steering*. Counter-steering is a well-known method for steering motorcycles. A turn is initiated by turning the steering handle in the opposite direction (to the intended direction of the turn). This leans the bicycle into the turn. After a certain point, because of the stability of the bicycle at speed, the steering angle turns back to the right angle for the turn to continue. Thus, the bicycle first turns in a direction opposite to the intended direction. Note that we assume the absence of dissipative forces acting on the bicycle.

To use this kind of behavior to initiate the turn, we specify a desired angle ϕ_d for the bicycle and rider and use the feedback linearized controller from Section 3.1.1. The effect of this controller is shown in Figure 8. Here, we start off by generating motion for the bicycle using a sinusoidal input as before. After a certain period of time, the sinusoidal input is stopped and the bicycle stabilizes to a straight forward motion. Now, the bicycle can be made to track a desired angular velocity. Given a forward velocity V and a desired angular velocity ω_d, the radius of the circular trajectory followed by the bicycle is $R_c = \frac{V}{\omega_d}$ and the lean angle ϕ_d can be easily determined. ϕ_d is given by setting the moment due to the weight of the bicycle about the point of contact with the ground equal to the moment due to the centrifugal force at the mass center of the bicycle. Thus,

$$m_t \omega_d^2 R_c l_t \cos\phi_d = m_t g l_t \sin\phi_d \tag{18}$$

Hence

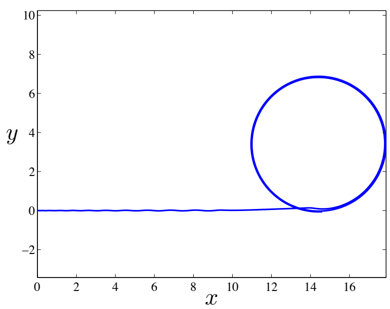
$$\phi_d = \arctan\frac{\omega^2 R_c}{g}. \tag{19}$$

Here m_t is the total mass of the bicycle and l_t is the (approximate) distance of the center of mass of the bicycle and rider above the ground in the upright position. In Figure 8, note the non-zero roll angle required to keep the bicycle moving in a circle. Also note that $\rho = 0$ when the bicycle is traversing the circular trajectory, *i.e.* the rider stays fixed relative to the bicycle.

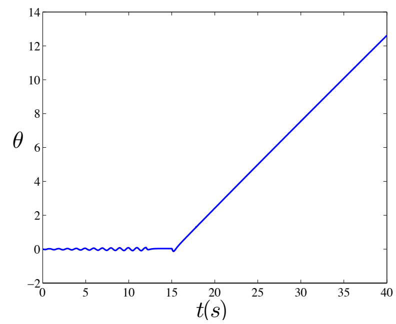
Counter-steering can be seen more clearly in a different case. Here, the goal is to converge to a trajectory with $(x_d, y_d, \theta_d) = (0, -3, 0)$, *i.e.* y_d = constant $= -3$. The controller used here is the same as in Eq. 19 except ω_d is now specified by the following control law

$$\omega_d = K_p^\omega (y_d - y_r) + K_d^\omega V \sin\theta_r. \tag{20}$$

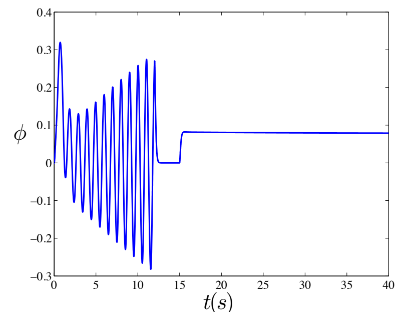
Figure 9 plots these results. The controller that converges on the desired trajectory is switched on at $t =$ 15s. It can be clearly seen from the plot of θ vs. t that the system turns initially in the counter-clockwise direction before turning back in the clockwise direction and converging on the desired trajectory. (Note that



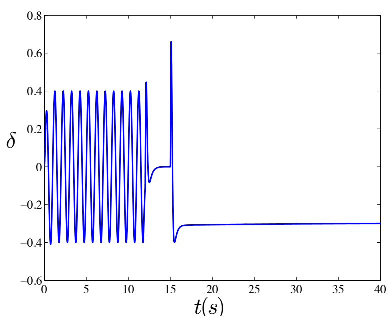
(a) TRAJECTORY OF CONTACT POINT OF REAR WHEEL



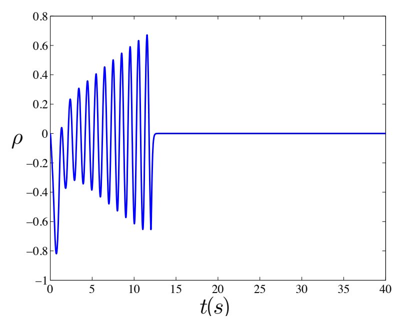
(b) θ VS t



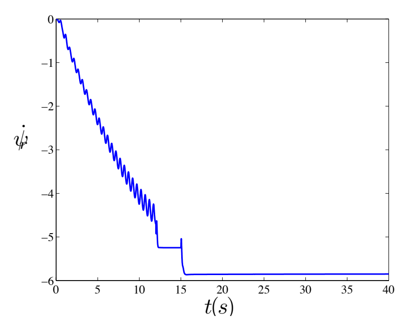
(c) ROLL ANGLE OF BICYCLE



(d) STEERING ANGLE

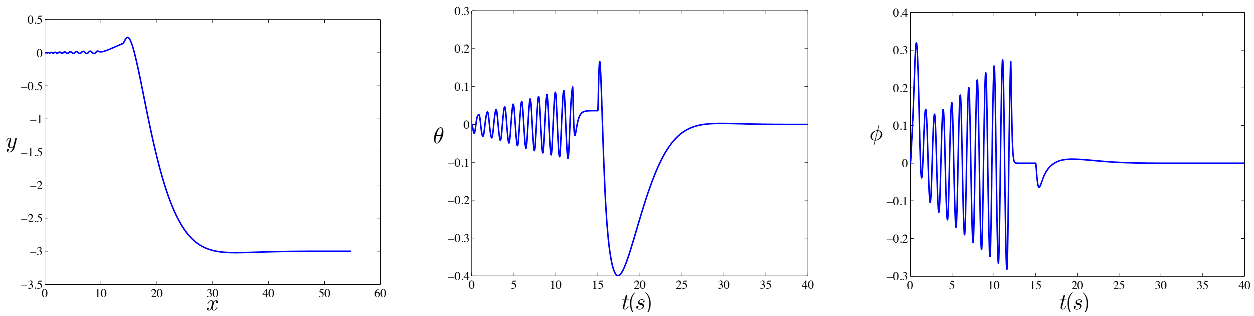


(e) LEAN ANGLE



(f) ANGULAR VELOCITY OF REAR WHEEL

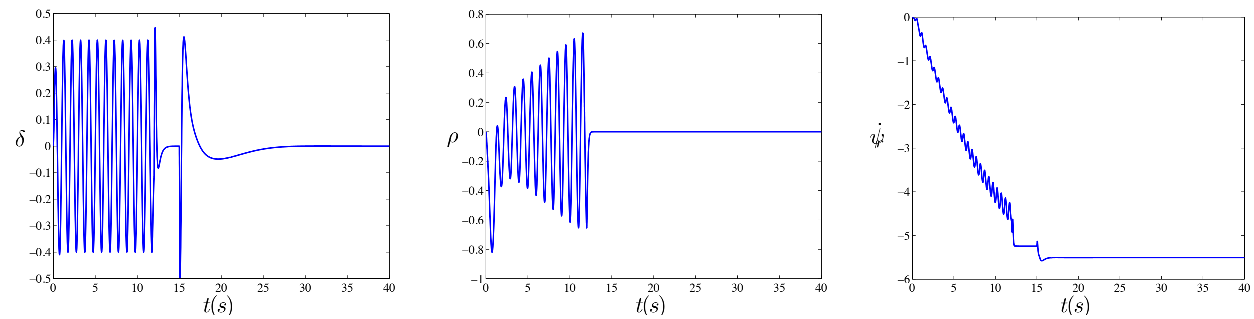
Figure 8: SIMULATION RESULTS FOR TRACKING A CIRCULAR TRAJECTORY



(a) TRAJECTORY OF CONTACT POINT OF REAR WHEEL

(b) θ VS t

(c) ROLL ANGLE OF BICYCLE



(d) STEERING ANGLE

(e) LEAN ANGLE

(f) ANGULAR VELOCITY OF REAR WHEEL

Figure 9: SIMULATION RESULTS FOR LANE CHANGE

because of the convention adopted, positive δ is the result of turning the steering handle clockwise).

6 Conclusions

We have shown that it is possible to propel a pedal-less bicycle solely by periodic motion of the steering handle of the bicycle. We have also shown that a rider on the bicycle can actively balance the bicycle while propelling the bicycle forward in this manner. We have also shown how, once the bicycle gains some speed, the bicycle can be controlled to converge to a desired trajectory. The analysis presented here offers the first explanation of our original observation that bicycle riders can sometimes balance their bicycles almost at rest.

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